

An Adaptive Method for Valuing an Option on Assets with Uncertainty in Stochastic Volatility

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Abstract

We present an adaptive approach for valuing the European call option on assets with stochastic volatility. The essential feature of the method is a reduction of uncertainty in latent volatility due to a Bayesian learning procedure. Starting from a discrete-time stochastic volatility model, we derive a recurrence equation for the variance of the innovation term in latent volatility equation. This equation describes a reduction of uncertainty in volatility which is crucial for option pricing. To implement the idea of adaptive control, we use the risk-minimization procedure involving random volatility with uncertainty. By using stochastic dynamic programming and a Bayesian approach, we derive a recurrence equation for the risk inherent in writing the option. This equation allows us to find the fair price of the European call option. We illustrate numerically that the adaptive procedure leads to a decrease in option price.

Keywords: Stochastic Volatility, Adaptive Decision Process, Bellman's equation.

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1 Introduction

Empirical observations on derivative prices show that implied volatilities vary with strike price giving the well known volatility “smile” effect [1]. This suggests that the behavior of the asset price, on which the option is written, may be captured by models that recognize the stochastic nature of volatility (SV). One can describe the underlying stock by a random process that is driven by a random volatility (see, for example, [2, 3, 4] and references therein). A common feature of these models is that the random volatility is described by a random process with *known* statistical characteristics. The problem with this approach is that the volatility is not a tradeable asset, and therefore the classical hedging can not be applied. As a result the equation for an option price involves an unknown parameter, the market price of volatility risk [2]. The difficulty is that this parameter is not directly observable and one has to make additional assumptions on the pricing of volatility risk. Moreover, in practice it is very difficult to suggest the *exact* statistics for the volatility in advance. A different approach to stochastic volatility has been suggested in [5, 6]. Instead of finding the exact option price one can focus on the pricing bands for options. The other problem of stochastic volatility models is associated with the efficient estimation of the *unobserved* volatility process from financial data. These lead some researchers to accept the idea of *uncertain* volatility when all prices for the option are possible within some range [7]. The question arises whether this uncertainty can be reduced during decision making. One of the main purposes of this work is to answer this question by using the simplest SV-model, the idea of Bayesian learning procedure and adaptive decision process (see [8, 9]). We suppose that some of the statistical properties of volatility are not known initially. Instead, we assume that we have an *a priori* estimation for them. By using the Bayesian approach (see [9]), we revise these *a priori* characteristics of random volatility. To implement the idea of adaptive feedback control for option pricing, we use a risk-minimization procedure (see [10] and references therein) and stochastic dynamic programming [8, 11, 12]. This work extends the idea of using adaptive processes in option pricing suggested in [13]. Application of stochastic dynamic programming for pricing of derivatives can be found in [14, 15]. It should be noted that the Bayesian learning approach to option pricing was also used in [16, 17, 18, 19] in different contexts. Bayesian estimation of a stochastic volatility model by using option price was proposed in [20]. The outline of the paper is as follows. In Section 2, we introduce a discrete-time stochastic volatility model and describe the Bayesian learning procedure. We derive the recurrence equation for the variance of the innovation term in latent volatility equation. In Section 3, we describe the risk-minimization procedure and

derive the Bellman's equation for the risk inherent in writing the option. By using this equation we find the fair price of European call option. We illustrate numerically that the adaptation procedure leads to a decrease in the option price.

2 Uncertain Stochastic Volatility and Adaptation

In this paper we consider a simple market with two traded assets: a riskless bond, B_n , and a risky asset (stock), S_n , evolving at discrete times $n = 0, 1, \dots, N$. The bond price B_n is governed by the recurrence relation

$$B_{n+1} = (1 + r)B_n, \quad B_0 > 0, \quad (1)$$

with the constant interest rate $r > 0$. The stock price S_n is governed by the stochastic difference equation

$$\ln \left(\frac{S_{n+1}}{S_n} \right) = \xi_n, \quad S_0 > 0, \quad (2)$$

where the stochastic return ξ_n is modelled as follows

$$\xi_n = \mu + \sigma \delta_n e^{h_n/2}. \quad (3)$$

Here μ is the mean return from holding a stock at time n , σ is the instantaneous volatility, h_n is the log-volatility (latent volatility) at time n that follows a stationary AR(1)-process:

$$h_{n+1} = \varphi h_n + u_n, \quad (4)$$

where φ is the ‘‘persistence’’ parameter of volatility: $|\varphi| < 1$. This is the simplest version of a stochastic volatility model and gives a discrete-time approximation for standard continuous stochastic volatility models (see, for example, [4, 21]). There are two sources of uncertainty in stochastic equations (3) and (4), namely the innovation terms δ_n and u_n . We assume that δ_n and u_n are both Gaussian sequences of mutually independent random variables, and δ_n has the following probability density function:

$$\varphi(\delta) = \frac{d}{d\delta} \mathbf{P} \{ \delta_n < \delta \} = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{\delta^2}{2} \right\}. \quad (5)$$

Let us now discuss the statistical properties of u_n . The empirical observations suggest that the log-volatility can be reasonably approximated by the Gaussian distribution [22, 23, 24]. Suppose that the investor does not know the exact value of the variance of u_n . With enough

information from the past history, the investor is assumed to have an *a priori* value for it, such that the probability density function for the first term, u_0 , is

$$p_0(u) = \frac{d}{du} \mathbf{P} \{u_0 < u\} = \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp \left\{ -\frac{u^2}{2\sigma_0^2} \right\}. \quad (6)$$

Our idea is to use an adaptive procedure by which the uncertainty regarding u_n can be reduced by using the equations (2)-(4). For this purpose one needs an equation involving both random sequences, u_n and δ_n in a linear combination. From (2)-(4) one can find

$$\ln S_{n+1} + h_{n+1} = \ln S_n + \mu + \sigma\delta_n e^{h_n/2} + \varphi h_n + u_n. \quad (7)$$

If we start with the given values of S_0 and h_0 , it follows from (5) and (7) that the likelihood of $\ln S_1 + h_1$ conditional on $u_0 = u$ is

$$L(\ln S_1 + h_1|u) = C_L \exp \left\{ -\frac{(\ln(\frac{S_1}{S_0}) + h_1 - \mu - \varphi h_0 - u)^2}{\sigma^2 e^{h_0}} \right\}, \quad (8)$$

where C_L is independent from u . By using (6) and Bayes' rule

$$p_1(u|\ln S_1 + h_1) = \frac{L(\ln S_1 + h_1|u) p_0(u)}{\int L(\ln S_1 + h_1|u) p_0(u) du} \quad (9)$$

(see [9]) one can find *a posteriori* pdf of u_1 conditional on $\ln S_1 + h_1$:

$$p_1(u|\ln S_1 + h_1) = C_1 \exp \left\{ -\frac{\left(\ln(\frac{S_1}{S_0}) + h_1 - \mu - \varphi h_0 - u\right)^2}{\sigma^2 e^{h_0}} \right\} \exp \left\{ -\frac{u^2}{2\sigma_0^2} \right\}, \quad (10)$$

where C_1 is independent of u . Equation (10) gives the learning procedure that can be used at each stage of the process to revise the probability density function for u_n . By using (10) we can find the recurrence relation for $p_n(u)$:

$$p_{n+1}(u) = C_{n+1} \exp \left\{ -\frac{\left(\ln(\frac{S_{n+1}}{S_n}) + h_{n+1} - \mu - \varphi h_n - u\right)^2}{\sigma^2 e^{h_n}} \right\} p_n(u). \quad (11)$$

It is a well known property of Gaussian distribution (see [9]) that this learning procedure gives a revised probability density function that is also Gaussian:

$$p_n(u) = \frac{d}{du} \mathbf{P} \{u_n < u\} = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp \left\{ -\frac{(u - m_n)^2}{2\sigma_n^2} \right\}. \quad (12)$$

The standard deviation σ_n and the mean m_n at successive stages are given by recurrence equations

$$\sigma_{n+1}^2 = \frac{\sigma^2}{e^{-h_n}\sigma_n^2 + \sigma^2}\sigma_n^2, \quad (13)$$

$$m_{n+1} = \frac{\sigma^2}{e^{-h_n}\sigma_n^2 + \sigma^2}m_n + \frac{\sigma_n^2[\ln(\frac{S_{n+1}}{S_n}) - \mu + h_{n+1} - \varphi h_n]}{\sigma_n^2 + \sigma^2 e^{h_n}}. \quad (14)$$

At each discrete time n , the uncertainty about the value of u_n is described by the probability density function $p_n(u)$ given by (12) which is completely specified by the mean value m_n and the standard deviation σ_n . These state variables are the *sufficient statistics*, and their transformation from one stage to the next is given by equations (13) and (14). Equation (13) shows that at every stage

$$\sigma_{n+1}^2 < \sigma_n^2,$$

that is, the variance of u_{n+1} is smaller than the variance of u_n . In other words the uncertainty about the innovation u_n is reduced at every stage n . This is crucial for option pricing. Let us note that although $\sigma_n^2 \rightarrow 0$ as $n \rightarrow \infty$, stochastic volatility does not disappear overall. Now we are in a position to apply the adaptive procedure (13) for the pricing of an European call option.

3 Adaptive stochastic optimization

Assume that an investor sells a European call option with strike price X for C_0 and invests the money in a portfolio containing Δ_0 shares and θ_0 bonds. The investor is concerned with hedging this position. It is well known that in incomplete markets a portfolio replicating the payoff of the option ceases to exist. Therefore the investor tries to find a trading strategy that reduces the risk of an option position to some intrinsic value. The value of the portfolio V_n at time n is given by

$$V_n = \Delta_n S_n + \theta_n B_n, \quad V_0 = C_0. \quad (15)$$

Using the self-financed trading strategy condition

$$(\Delta_{n+1} - \Delta_n)S_{n+1} + (\theta_{n+1} - \theta_n)B_{n+1} = 0,$$

one can obtain an equation for V_n :

$$V_n = (1 + r)V_{n-1} + \Delta_{n-1}(e^{\mu + \sigma \delta_{n-1} e^{\frac{h_{n-1}}{2}}} - 1 - r)S_{n-1}. \quad (16)$$

Let us recall the theory of risk-minimization in option pricing that was developed in [25, 26, 27] (see also [28, 29, 30]). The investor's purpose is to choose a trading strategy $\{\Delta_0, \dots, \Delta_{N-1}\}$

such that the terminal value of the portfolio, V_N , should be as close as possible to the options payoff: $(S_N - X, 0)^+$. Thus, the expected value of their difference, under the "real-world" probability measure, must be equal to zero: $\mathbf{E}\{(S_N - X, 0)^+ - V_N\} = 0$, while the variance

$$R = \mathbf{E}\{((S_N - X, 0)^+ - V_N)^2\} \quad (17)$$

as a measure of the risk should be minimized. Let us consider the problem of minimizing the risk function R for an N -stage process, starting from the initial states

$$S_0 = S, \quad V_0 = V, \quad h_0 = h \quad (18)$$

with *a priori* probability density $p_0(u)$ specified by the mean $m_0 = 0$, and the standard deviation $\sigma_0 = \sigma_u$. Here we use a stochastic programming procedure proposed in [13]. Let us introduce the minimal risk

$$R_N(S, V, h, \sigma_u) = \min_{\Delta_0, \dots, \Delta_{N-1}} \mathbf{E}\{((S_N - X, 0)^+ - V_N)^2\} \quad (19)$$

that can be achieved by starting from the initial state (18) with a priori pdf $p_0(u)$. After the first decision $\Delta_0 = \Delta$ of the N -stage process we have

$$S_1 = S e^{\mu + \sigma \delta e^{h/2}}, \quad (20)$$

$$V_1 = (1 + r)V + \Delta(e^{\mu + \sigma \delta e^{h/2}} - 1 - r)S, \quad (21)$$

$$h_1 = \varphi h + u, \quad (22)$$

$$\sigma_1^2 = \frac{\sigma^2 \sigma_u^2}{e^{-h} \sigma_u^2 + \sigma^2}. \quad (23)$$

The principal of optimality yields the general functional recurrence equation (Bellman's equation)

$$R_N(S, V, h, \sigma_u) = \min_{\Delta} \mathbf{E} \left\{ R_{N-1}(S e^{\mu + \sigma \delta e^{h/2}}, \right. \quad (24)$$

$$\left. (1 + r)V + \Delta(e^{\mu + \sigma \delta e^{h/2}} - 1 - r)S, \varphi h + u, \frac{\sigma^2 \sigma_u^2}{e^{-h} \sigma_u^2 + \sigma^2}) \right\}.$$

(see [8, 11, 12]). By using the explicit expressions for $\varphi(\delta)$ and $p_0(u)$, the equation (24) can be rewritten as follows

$$R_N(S, V, h, \sigma_u) = \min_{\Delta} \left\{ \frac{1}{2\pi\sigma_u} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{N-1}(S e^{\mu + \sigma \delta e^{h/2}}, \right. \quad (25)$$

$$\left. (1 + r)V + \Delta(e^{\mu + \sigma \delta e^{h/2}} - 1 - r)S, \varphi h + u, \frac{\sigma^2 \sigma_u^2}{e^{-h} \sigma_u^2 + \sigma^2}) e^{-\frac{\delta^2}{2} - \frac{u^2}{2\sigma_u^2}} d\delta du \right\}.$$

To solve (25), we need to know the value of the risk function $R_N(S, V, h, \sigma_u)$ for $N = 1$. It follows from (17) that

$$R_1(S, V, h) = \min_{\Delta} \mathbf{E} \left\{ ((Se^{\mu+\sigma\delta e^{h/2}} - X, 0)^+ - (1+r)V - \Delta(e^{\mu+\sigma\delta e^{h/2}} - 1 - r)S)^2 \right\}. \quad (26)$$

Using (5) we have

$$R_1(S, V, h) = \min_{\Delta} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ((Se^{\mu+\sigma\delta e^{h/2}} - X, 0)^+ - (1+r)V - \Delta(e^{\mu+\sigma\delta e^{h/2}} - 1 - r)S)^2 e^{-\frac{\delta^2}{2}} d\delta \right\}, \quad (27)$$

where

$$(Se^{\mu+\sigma\delta e^{h/2}} - X, 0)^+ = \begin{cases} Se^{\mu+\sigma\delta e^{h/2}} - X & \text{for } \sigma^{-1}e^{-h/2}(\ln(XS^{-1}) - \mu) < \delta \\ 0 & \text{otherwise} \end{cases}. \quad (28)$$

The integral in (27) can be evaluated exactly (see Appendix A). This allows us to find the explicit expressions for the optimal policy, $\Delta_1(S, V, h)$ and the risk $R_1(S, V, h)$ when there is one stage-to-go (see Appendix A). Now putting $N = 2$ in equation (25) and using the expression for R_1 , one can find the risk function R_2 :

$$R_2(S, V, h) = \min_{\Delta} \left\{ \frac{1}{2\pi\sigma_u} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_1(Se^{\mu+\sigma\delta e^{h/2}}, (1+r)V + \Delta(e^{\mu+\sigma\delta e^{h/2}} - 1 - r)S, \varphi h + u) e^{-\frac{\delta^2}{2} - \frac{u^2}{2\sigma_u^2}} d\delta du \right\}$$

Note that at each stage, the risk function R_n does not depend on the mean m_n . What is more, adaptation procedure starts only at the third step, when the risk R_3 becomes a function of σ_u that is $R_3 = R_3(S, V, h, \sigma_u)$. The procedure can be repeated any number of times to give the solution of the problem for any value of N . The attractive feature of this algorithm is the simplicity with which the adaptation procedure can be applied. The initial investment V determining a fair option price $C = V$ can be obtained from the equation

$$\frac{\partial R_N(S, V, h, \sigma_u)}{\partial V} = 0. \quad (29)$$

In particular, for a one stage process ($N = 1$), after minimization, we obtain

$$V(S, h) = \frac{1}{2\sqrt{\pi}(1+r)} e^{-\mu - \frac{e^h \sigma^2}{2}} \left(\sqrt{2} e^{\mu + \frac{3e^h \sigma^2}{2}} + e^{e^{\mu} + \frac{e^h \sigma^2}{2}} \sqrt{\pi} \left(\ln\left(\frac{X}{S}\right) + S(1 - 2r + 2\mu) + (\mu + e^h \sigma^2 - \ln(\frac{X}{S}))(2\mathcal{N}(d) - 1) \right) \right), \quad (30)$$

where $\mathcal{N}(d)$ is the cumulative distribution function for a Gaussian variable:

$$\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{s^2}{2}} ds, \quad d = \frac{(\mu + e^h \sigma^2 - \ln(\frac{X}{S}))e^{-h/2}}{\sigma}.$$

The above results can be compared to those corresponding to the standard model without an adaptive procedure. In the later case, an *a priori* density function for u (6) is kept at each stage, and the risk function R_N becomes the function of S, V , and h only. The Bellman recurrence equation for the risk minimization problem is then given by

$$R_N(S, V, h) = \min_{\Delta} \left\{ \frac{1}{2\pi\sigma_u} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_{N-1}(Se^{\mu+\sigma\delta e^{h/2}}, (1+r)V + \Delta(e^{\mu+\sigma\delta e^{h/2}} - 1 - r)S, \varphi h + u) e^{-\frac{\delta^2}{2} - \frac{u^2}{2\sigma_u^2}} d\delta du \right\}, \quad (31)$$

where R_1 is the same as (27). To illustrate our adaptive control method we value the European call option with the strike price $X = 50$, the initial log-volatility value $h = 0.1$, the interest rate $r = 0.05$, the expected return $\mu = 0.1$, the volatility parameter $\sigma = 0.2$, the maturity of the option $T = 1$, and $\varphi = 0.1$. We also calculate the option price for the constant volatility case ($h = 0$). Figure 1 shows the results for the option price as a function of S for different number of steps of the adaptive (learning) procedure. To illustrate the usefulness the adaptive approach, we computed the value of a European call option for the standard (no-learning) procedure using equation (31). In Figure 2 we show the difference between the option prices with and without adaptation. The number of steps $N = 12$. It is clear that the adaptive procedure leads to a decrease in option price. Let us now discuss the “smile” effect arising from the above methodology, that is, implied volatility varies with strike price. Using equations (24) and (29) for $N = 12$, we can retrieve various call option prices, $C_{adap}(X)$, for different strike prices X . Implied volatility $I = \sigma(X)$, can then be computed by inverting the formula

$$C_{BS}(X, I) = C_{adap}(X), \quad (32)$$

where C_{BS} is the usual Black-Scholes price. We plot our results in Figure 3, where we get a “smile” curve, with a minimum near at-the-money. Thus, the adaptive methodology presented here is shown to reproduce the “smile” effect observed in the market. There is a one-to-one relationship between the volatility “smile” and the implied distribution. The volatility “smile” in Figure 3 implies some kurtosis and skewness in the implied distribution.

4 Conclusions

In contrast to most stochastic volatility models we applied an adaptive control procedure which allows us to revise the stochastic characteristics of latent volatility during decision making. We assumed that the statistical properties of an innovation term in a log-volatility

equation are not known initially, but instead we have an a priori estimation for them. By using Bayesian analysis, we derived the recurrence equation for the variance of innovation term. This equation describes a reduction of uncertainty about volatility which is crucial for option pricing. We implemented the idea of adaptive procedure by using the risk-minimization analysis and stochastic dynamic programming. We showed that the adaptation leads to a decrease in the option price compared to the standard models without learning. The adaptive algorithm allows the investor to hedge his position in a consistent way between two extremes: a completely uncertain volatility and an ideal situation of known volatility. Of course this paper leaves many open questions how to model the uncertainties in a stochastic volatility setting. For example, one can introduce the uncertainty in the stochastic return (3) rather than in the latent volatility (4). We hope to address these questions in the future works.

Appendix A

To evaluate the integral in (27), we need to split it into two integrals. That is,

$$R_1(S, V, h) = \min_{\Delta} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\sigma^{-1}e^{-h/2}(\ln(XS^{-1})-\mu)} (-(1+r)V - \Delta(e^{\mu+\sigma\delta e^{h/2}} - 1 - r)S)^2 e^{-\frac{\delta^2}{2}} d\delta + \right. \\ \left. \frac{1}{\sqrt{2\pi}} \int_{\sigma^{-1}e^{-h/2}(\ln(XS^{-1})-\mu)}^{\infty} \left(Se^{\mu+\sigma\delta e^{h/2}} - X - (1+r)V - \Delta(e^{\mu+\sigma\delta e^{h/2}} - 1 - r)S \right)^2 e^{-\frac{\delta^2}{2}} d\delta \right\}. \quad (\text{A-1})$$

By using *Mathematica* one can get the following expression for R_1 :

$$R_1(S, V, h) = \min_{\Delta} \left\{ \frac{e^{-l}}{2\sqrt{2\pi}} \left(2e^{\mu+\frac{3e^h\sigma^2}{2}} S(1+\Delta)(-2(1+r)V + \ln(\frac{X}{S})\Delta + \right. \right. \\ \left. S(1+\mu+\Delta(e^h\sigma^2-2r+\mu)))\sigma + ((V+\ln(\frac{X}{S})+S(-1+r\Delta-(1+\Delta)\mu))^2 + e^h S(1+\Delta)^2\sigma^2)\mathcal{N}(d) \right) + \\ \left. \frac{e^l}{2\sqrt{2\pi}} \left(((-2e^{\mu+\frac{3e^h\sigma^2}{2}} S\Delta(-2(1+r)V + \Delta(\ln(\frac{X}{S}) + S(-1+\mu))))\sigma + \right. \right. \\ \left. \left. \sqrt{2\pi}(((1+r)V + S\Delta(\mu + e^h\sigma^2))^2 + e^h S^2\Delta^2\sigma^2)(2 - 2\mathcal{N}(d)) \right) \right) \quad (\text{A-2})$$

where

$$\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-\frac{s^2}{2}} ds, \quad d = \frac{(\mu + e^h\sigma^2 - \ln(\frac{X}{S}))e^{-h/2}}{\sigma}, \quad l = \mu + \frac{e^h\sigma^2}{2}$$

($\mathcal{N}(d)$ is the cumulative distribution function for the normal distribution). Differentiation with respect to Δ leads to the optimal first decision when there is one stage-to-go, $\Delta_1(S, V, h)$, starting from the initial state S and V :

$$\Delta_1(S, V) = \left(e^{-l} \left(2S \left(e^{\mu+\frac{3e^h\sigma^2}{2}} S(r-\mu)\sigma e^{h/2} - e^l \sqrt{2\pi}((r-\mu)((1+\mu)V + \ln(\frac{X}{S}) - \mu + e^h\sigma^2) \right. \right. \right. \\ \left. \left. \left. e^h S\sigma^2 \right) \right) + 2e^l \sqrt{2\pi} S((r-\mu)(S - \ln(\frac{X}{S}) + S\mu) - e^h S\sigma^2)(2\mathcal{N}(d) - 1) \right) \\ \left. (\sqrt{2\pi} S^2((r^2 + e^h\sigma^2))^{-1}, \right.$$

and substituting this in the expression for $R_1(S, V, h)$ gives

$$R_1(S, V, h) = \frac{e^{-l}}{2\sqrt{2\pi}} \left(2e^{\mu+\frac{e^h\sigma^2}{2}} S((1+r)V - \ln(\frac{X}{S}) + S(1+2\mu)\sigma + \right. \\ \left. e^l \sqrt{2\pi})(V + rV + \ln(\frac{X}{S}) + S(-1+r+\mu))^2 + e^h S^2(1+r)^2\sigma^2 + \right. \\ \left. e^l \sqrt{2\pi} \left(((1+r)V + S(r-\mu))^2 + e^h S^2\sigma^2 + ((S - \ln(\frac{X}{S})) \right. \right. \\ \left. \left. (-2(1+r)V - \ln(\frac{X}{S}) + e^{h/2} S(1-2r+2\mu)) + e^h S^2(1+\sigma^2)2\mathcal{N}(d) \right) \right).$$

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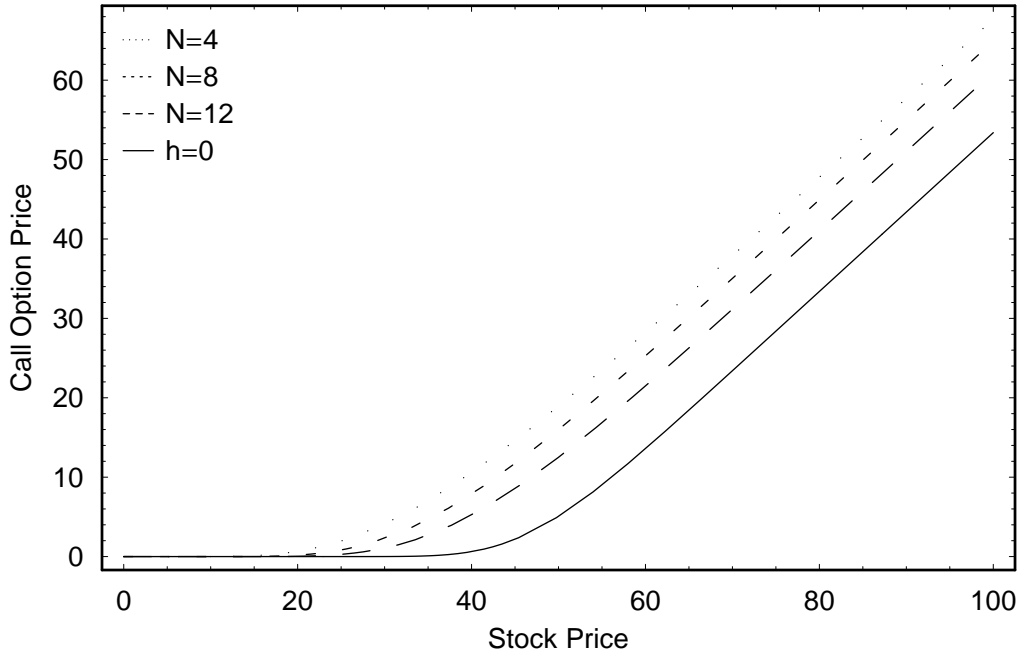


Figure 1: *Option price as a function of asset price for different number of stages of the adaptive process.*

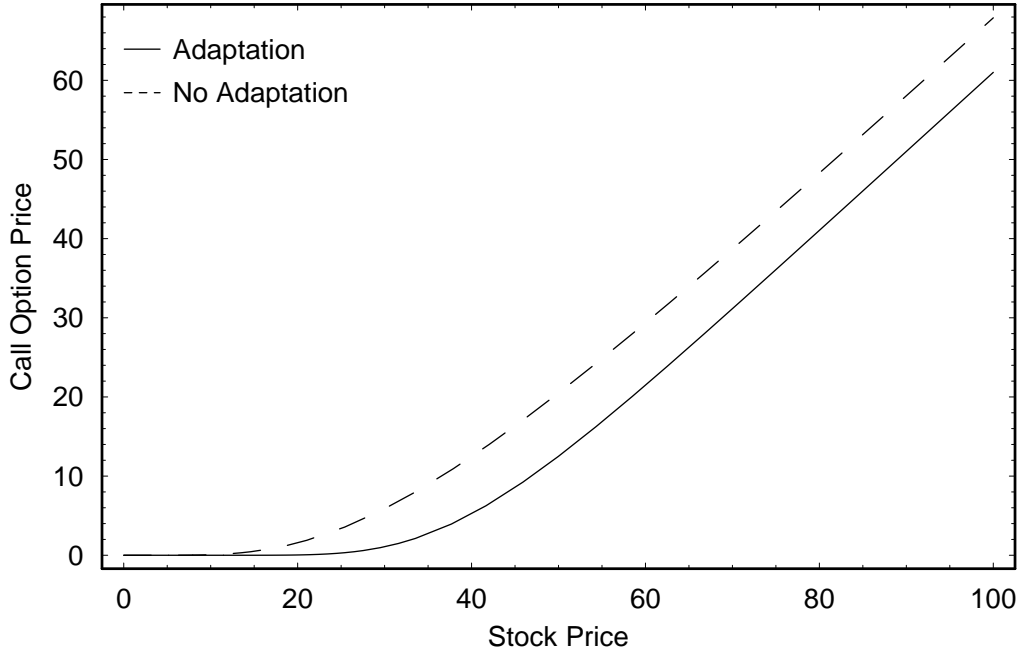


Figure 2: *Comparison of option price with and without adaptation for $N = 12$.*

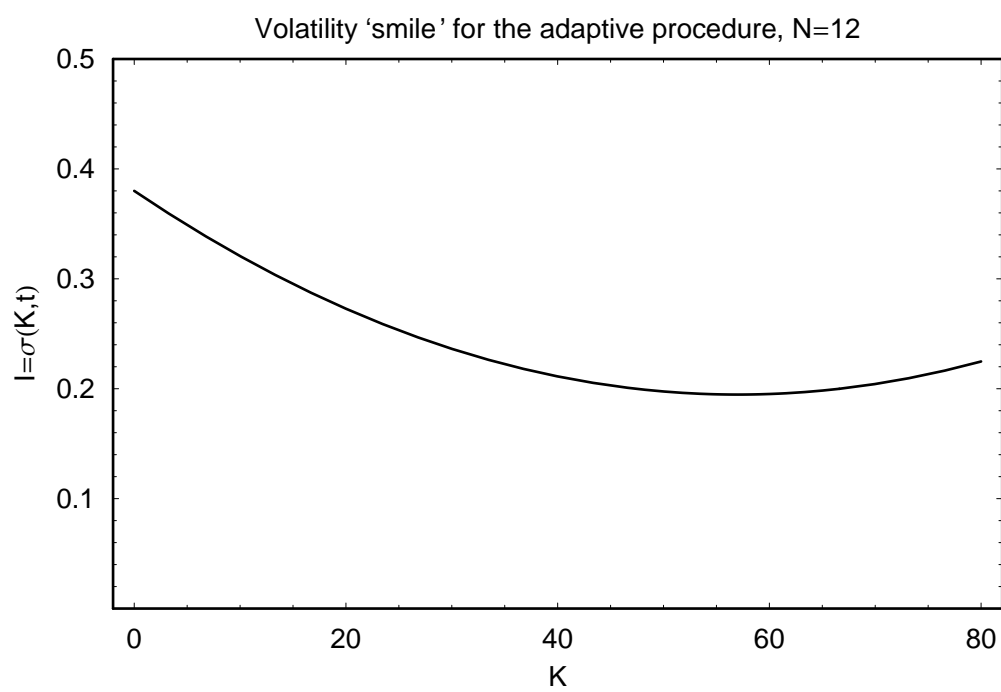


Figure 3: *Implied volatility $I = \sigma(X)$ as a function of strike price X . The constants are taken as in Figures 1 and 2.*